Divergence Theorem

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Overview

The divergence form of **Green's Theorem** in the plane states that the net outward flux of a vector field across a simple closed curve can be calculated by integrating the divergence of the field over the region enclosed by the curve.

The corresponding theorem in three dimensions, called the **Divergence Theorem**, states that the net outward flux of a vector field across a closed surface in space can be calculated by integrating the divergence of the field over the region enclosed by the surface.

Mikhail Vassilievich Ostrogradsky (1801-1862) was the first mathematician to publish a proof of the Divergence Theorem.

Upon being denied his degree at Kharkhow Univesity by the minister for religious affairs and national education (for atheism), Ostrogradsky left Russia for Paris in 1822, attracted by the presence of Laplace, Legendre, Fourier, Poisson, and Cauchy.

While working on the theory of heat in the mid-1820s, he formulated the Divergence Theorem as a tool for converting volume integrals to surface integrals.

Cal Fridrich Gauss

Cal Fridrich Gauss (1777-1855) has already proved the theorem while working on the theory of gravitation, but his notebooks were not to be publised until many years later.

The theorem is sometimes called Gauss's theorem.

The list of Gauss's accomplishments in science and mathematicss is truly astonishing, ranging from the invention of the electric telegraph (with Wilhelm Weber in 1833) to the development orbits and to work in non-Euclidean geometry that later became fundamental to Einstein's general theory of relativity.

Divergence in Three Dimensions

The divergence of a vector field $\mathrm{F} = M(x, y, z)\mathrm{i} + N(x, y, z)\mathrm{j} + P(x, y, z)\mathrm{k}$ is the scalar function

$$\mathrm{divF} = \nabla.\mathrm{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}.$$

The symbol "div **F**" is read as "divergence of **F**" or "div **F**." The notation $\nabla.\mathrm{F}$ is read "del dot **F**."

Div **F** has the same physical interpretation in three dimensions that it does in two. If F is the velocity field of a fluid flow, the value of div F at a point (x, y, z) is the rate at which fluid is being piped in or drained away at (x, y, z).

The divergence is the flux per unit volume or flux density at the point.

The Divergence Theorem says that under suitable condition the outward flux of a vector field across a closed surface (oriented outward) equals the tiple integral of the divergence of the field over the region enclosed by the surface.

The Divergence Theorem

The flux of a vector field $\mathsf{F} = M\mathsf{i} + N\mathsf{j} + P\mathsf{k}$ across a closed oriented surface S in the direction of the surface's outward unit normal field n equals the integral of $\nabla.\mathsf{F}$ over the region D enclosed by the surface :

$$\iint_S \mathsf{F}.\mathsf{n}\ d\sigma = \iiint_D \nabla.\mathsf{F}\ dV.$$

Exercise 1.

Find the divergence of the gravitational field $\mathsf{F} = -\frac{GM(x\mathsf{i}+y\mathsf{j}+z\mathsf{k})}{(x^2+y^2+z^2)^{3/2}}$.

Solution for Exercise 1 :

$$\mathbf{F} = -\frac{GM(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})}{(x^2+y^2+z^2)^{3/2}}$$

$$\Rightarrow div\mathbf{F} = -GM\left[\frac{(x^2+y^2+z^2)^{3/2}-3x^2(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3}\right]$$

$$-GM\left[\frac{(x^2+y^2+z^2)^{3/2}-3y^2(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3}\right]-GM\left[\frac{(x^2+y^2+z^2)^{3/2}-3z^2(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3}\right]$$

$$= -GM\left[\frac{3(x^2+y^2+z^2)^2-3(x^2+y^2+z^2)(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{7/2}}\right] = 0$$

Exercise 2.

Use the Divergence Theorem to find the outward flux of F *across the boundary of the region D.*

(a) Cube : F=*(y-x)*i+*(z-y)*j+*(y-x)*k
D : *The cube bounded by the planes* $x = \pm 1$, $y = \pm 1$, *and* $z = \pm 1$

(b) $\mathrm{F} = x^2\mathrm{i} + y^2\mathrm{j} + z^2\mathrm{k}$

  (a) Cube : D : *The cube cut from the first octant by the planes* $x = 1$, $y = 1$, *and* $z = 1$

  (b) Cube : D : *The cube bounded by the planes* $x = \pm 1$, $y = \pm 1$, *and* $z = \pm 1$

  (c) Cylindrical can : D : *The region cut from the solid cylinder* $x^2 + y^2 \leq 4$ *by the planes* $z = 0$ *and* $z = 1$

Solution for Exercise 2

1. $\frac{\partial}{\partial x}(y-x) = -1,\ \frac{\partial}{\partial y}(z-y) = -1,\ \frac{\partial}{\partial z}(y-x) = 0 \Rightarrow \nabla \cdot \mathbf{F} = -2 \Rightarrow$ Flux $= \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} -2dxdydz = -2(2^3) = -16$
2. $\frac{\partial}{\partial x}(x^2) = 2x,\ \frac{\partial}{\partial y}(y^2) = 2y,\ \frac{\partial}{\partial z}(z^2) = 2z \Rightarrow \nabla \cdot \mathbf{F} = 2x + 2y + 2z$
   (a) Flux $= \int_0^1\int_0^1\int_0^1 (2x+2y+2z)dxdydz =$
   $\int_0^1\int_0^1 \left[x^2 + 2x(y+z)\right]_0^1 dydz = \int_0^1\int_0^1 (1+2y+2z)dydz =$
   $\int_0^1 \left[y(1+2z) + y^2\right]_0^1 dz = \int_0^1 (2+2z)dz = \left[2z + z^2\right]_0^1 = 3$
   (b) Flux $= \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} (2x+2y+2z)dxdydz =$
   $\int_{-1}^{1}\int_{-1}^{1} \left[x^2 + 2x(y+z)\right]_{-1}^{1} dydz = \int_{-1}^{1}\int_{-1}^{1} (4y+4z)dydz =$
   $\int_{-1}^{1} \left[2y^2 + 4yz\right]_{-1}^{1} dz = \int_{-1}^{1} 8zdz = \left[4z^2\right]_{-1}^{1} = 0$
   (c) In cylindrical coordinates, Flux $= \iiint\limits_D (2x+2y+2z)dxdydz$
   $= \int_0^1\int_0^{2\pi}\int_0^2 (2r\cos\theta + 2r\sin\theta + 2z)rdrd\theta dz =$
   $\int_0^1\int_0^{2\pi} \left[\frac{2}{3}r^3\cos\theta + \frac{2}{3}r^3\sin\theta + zr^2\right]_0^2 d\theta dz$
   $= \int_0^1\int_0^{2\pi} \left(\frac{16}{3}\cos\theta + \frac{16}{3}\sin\theta + 4z\right) d\theta dz =$
   $\int_0^1 \left[\frac{16}{3}\sin\theta - \frac{16}{3}\cos\theta + 4z\theta\right]_0^{2\pi} dz = \int_0^1 8\pi zdz = \left[4\pi z^2\right]_0^1 = 4\pi$

Exercise 3.

Use the Divergence Theorem to find the outward flux of F *across the boundary of the region* D.

1. Cylinder and paraboloid : $\mathrm{F} = y\mathrm{i} + xy\mathrm{j} - z\mathrm{k}$
   D : *The region inside the solid cylinder* $x^2 + y^2 \leq 4$ *between the plane* $z = 0$ *and the paraboloid* $z = x^2 + y^2$
2. Sphere : $\mathrm{F} = x^2\mathrm{i} + xz\mathrm{j} + 3z\mathrm{k}$
   D : *The solid sphere* $x^2 + y^2 + z^2 \leq 4$
3. Portion of sphere : $\mathrm{F} = x^2\mathrm{i} - 2xy\mathrm{j} + 3xz\mathrm{k}$
   D : *The region cut from the first octant by the sphere* $x^2 + y^2 + z^2 = 4$

Solution for Exercise 3

1. $\frac{\partial}{\partial x}(y) = 0$, $\frac{\partial}{\partial y}(xy) = x$, $\frac{\partial}{\partial z}(-z) = -1 \Rightarrow \nabla \cdot \mathbf{F} = x - 1$; $z = x^2 + y^2 \Rightarrow z = r^2$ in cylindrical coordinates $\Rightarrow$ Flux
$= \iiint_D (x-1)dzdydx = \int_0^{2\pi}\int_0^2\int_0^{r^2}(r\cos\theta - 1)dzrdrd\theta = \int_0^{2\pi}\int_0^2 (r^3\cos\theta - r^2)rdrd\theta$
$= \int_0^{2\pi}\left[\frac{r^5}{5}\cos\theta - \frac{r^4}{4}\right]_0^2 d\theta = \int_0^{2\pi}\left(\frac{32}{5}\cos\theta - 4\right)d\theta = \left[\frac{32}{5}\sin\theta - 4\theta\right]_0^2\pi = -8\pi$
2. $\frac{\partial}{\partial x}(x^2) = 2x$, $\frac{\partial}{\partial y}(xz) = 0$, $\frac{\partial}{\partial z}(3z) = 3 \Rightarrow \nabla \cdot \mathbf{F} = 2x + 3 \Rightarrow$ Flux $= \iiint_D (2x+3)dV$
$= \int_0^{2\pi}\int_0^{\pi}\int_0^2 (2\rho\sin\phi\cos\theta + 3)(\rho^2\sin\phi)d\rho d\phi d\theta =$
$\int_0^{2\pi}\int_0^{\pi}\left[\frac{\rho^4}{2}\sin\phi\cos\theta + \rho^3\right]_0^2 \sin\phi d\phi d\theta$
$= \int_0^{2\pi}\int_0^{\pi}(8\sin\phi\cos\theta + 8)\sin\phi d\phi d\theta = \int_0^{2\pi}\left[8\left(\frac{\phi}{2} - \frac{\sin 2\phi}{4}\right)\cos\theta - 8\cos\phi\right]_0^{\pi} d\theta =$
$\int_0^{2\pi}(4\pi\cos\theta + 16)d\theta = 32\pi$
3. $\frac{\partial}{\partial x}(x^2) = 2x$, $\frac{\partial}{\partial y}(-2xy) = -2x$, $\frac{\partial}{\partial z}(3xz) = 3x \Rightarrow$ Flux $= \iiint_D 3x dxdydz$
$= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^2 (3\rho\sin\phi\cos\theta)(\rho^2\sin\phi)d\rho d\phi d\theta = \int_0^{\pi/2}\int_0^{\pi/2} 12\sin^2\phi\cos\theta d\phi d\theta =$
$\int_0^{\pi/2} 3\pi\cos\theta d\theta = 3\pi$

Exercise 4.

Use the Divergence Theorem to find the outward flux of $\mathbf{F}$ *across the boundary of the region* D.

1. Cylindrical can : $\mathbf{F} = (6x^2 + 2xy)\mathbf{i} + (2y + x^2z)\mathbf{j} + 4x^2y^3\mathbf{k}$
   D : *The region cut from the first octant by the cylinder* $x^2 + y^2 = 4$ *and the plane* $z = 3$
2. Wedge : $\mathbf{F} = 2xz\mathbf{i} - xy\mathbf{j} - z^2\mathbf{k}$
   D : *The wedge cut from the first octant by the plane* $y + z = 4$ *and the elliptical cylinder* $4x^2 + y^2 = 16$
3. Sphere : $\mathbf{F} = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$
   D : *The solid sphere* $x^2 + y^2 + z^2 \leq a^2$

Solution for Exercise 4

1. $\frac{\partial}{\partial x}(6x^2+2xy) = 12x+2y$, $\frac{\partial}{\partial y}(2y+x^2z) = 2$,
$\frac{\partial}{\partial z}(4x^2y^3) = 0 \Rightarrow \nabla \cdot \mathbf{F} = 12x+2y+2 \Rightarrow \text{Flux} = \iiint\limits_D (12x+2y+2)dV$
$= \int_0^3 \int_0^{\pi/2} \int_0^2 (12r\cos\theta + 2r\sin\theta + 2)rdrd\theta dz$
$= \int_0^3 \int_0^{\pi/2} (32\cos\theta + \frac{16}{3}\sin\theta + 4)d\theta dz = \int_0^3 (32 + 2\pi + \frac{16}{3})dz = 112 + 6\pi$
2. $\frac{\partial}{\partial x}(2xz) = 2z$, $\frac{\partial}{\partial y}(-xy) = -x$, $\frac{\partial}{\partial z}(-z^2) = -2z \Rightarrow \nabla \cdot \mathbf{F} = -x \Rightarrow \text{Flux} = \iiint\limits_D -xdV$
$= \int_0^2 \int_0^{\sqrt{16-4x^2}} \int_0^{4-y} -xdzdydx = \int_0^2 \int_0^{\sqrt{16-4x^2}} (xy-4x)dydx =$
$\int_0^2 \left[\frac{1}{2}x(16-4x^2) - 4x\sqrt{16-4x^2}\right] dx$
$= \left[4x^2 - \frac{1}{2}x^4 + \frac{1}{3}(16-4x^2)^{3/2}\right]_0^2 = -\frac{40}{3}$
3. $\frac{\partial}{\partial x}(x^3) = 3x^2$, $\frac{\partial}{\partial y}(y^3) = 3y^2$, $\frac{\partial}{\partial z}(z^3) = 3z^2 \Rightarrow \nabla \cdot \mathbf{F} = 3x^2+3y^2+3z^2 \Rightarrow \text{Flux}$
$= \iiint\limits_D 3(x^2+y^2+z^2)dV$
$= 3\int_0^{2\pi} \int_0^{\pi} \int_0^a \rho^2(\rho^2 \sin\phi)d\rho d\phi d\theta = 3\int_0^{2\pi} \int_0^{\pi} \frac{a^5}{5}\sin\phi d\phi d\theta = 3\int_0^{2\pi} \frac{2a^5}{5}d\theta = \frac{12\pi a^5}{5}$

Exercise 5.

Use the Divergence Theorem to find the outward flux of F *across the boundary of the region D.*

1. Thick sphere : $\mathrm{F} = (x\mathrm{i} + y\mathrm{j} + z\mathrm{k})/\sqrt{x^2 + y^2 + z^2}$
   D : *The region* $1 \leq x^2 + y^2 + z^2 \leq 4$
2. Thick cylinder : $\mathrm{F} = \ln(x^2 + y^2)\mathrm{i} - \left(\frac{2z}{x}\tan^{-1}\frac{y}{x}\right)\mathrm{j} + z\sqrt{x^2 + y^2}\mathrm{k}$
   D : *The thick-walled cylinder* $1 \leq x^2 + y^2 \leq 2, \quad -1 \leq z \leq 2$

Solution for Exercise 5

1. Let $\rho = \sqrt{x^2+y^2+z^2}$. Then $\frac{\partial \rho}{\partial x} = \frac{x}{\rho}$, $\frac{\partial \rho}{\partial y} = \frac{y}{\rho}$,
$\frac{\partial \rho}{\partial z} = \frac{z}{\rho} \Rightarrow \frac{\partial}{\partial x}\left(\frac{x}{\rho}\right) = \frac{1}{\rho} - \left(\frac{x}{\rho^2}\right)\frac{\partial \rho}{\partial x} = \frac{1}{\rho} - \frac{x^2}{\rho^3}$. Similarly,
$\frac{\partial}{\partial y}\left(\frac{y}{\rho}\right) = \frac{1}{\rho} - \frac{y^2}{\rho^3}$ and $\frac{\partial}{\partial z}\left(\frac{z}{\rho}\right) = \frac{1}{\rho} - \frac{z^2}{\rho^3} \Rightarrow \nabla \cdot \mathbf{F} = \frac{3}{\rho} - \frac{x^2+y^2+z^2}{\rho^3} = \frac{2}{\rho}$
$\Rightarrow$ Flux
$= \iiint\limits_D \frac{2}{\rho}\, dV = \int_0^{2\pi}\int_0^{\pi}\int_1^2 \left(\frac{2}{\rho}\right)\left(\rho^2 \sin\phi\right) d\rho\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi} 3\sin\phi\, d\phi\, d\theta = \int_0^{2\pi} 6\, d\theta = 12\pi$

2. $\frac{\partial}{\partial x}[\ln(x^2+y^2)] = \frac{2x}{x^2+y^2}$, $\frac{\partial}{\partial y}\left(-\frac{2z}{x}\tan^{-1}\frac{y}{x}\right) = \left(-\frac{2z}{x}\right)\left[\frac{\left(\frac{1}{x}\right)}{1+\left(\frac{y}{x}\right)^2}\right] = -\frac{2z}{x^2+y^2}$,
$\frac{\partial}{\partial z}\left(z\sqrt{x^2+y^2}\right) = \sqrt{x^2+y^2}$
$\Rightarrow \nabla \cdot \mathbf{F} = \frac{2x}{x^2+y^2} - \frac{2z}{x^2+y^2} + \sqrt{x^2+y^2} \Rightarrow$ Flux
$= \iiint\limits_D \left(\frac{2x}{x^2+y^2} - \frac{2z}{x^2+y^2} + \sqrt{x^2+y^2}\right) dz\,dy\,dx$
$= \int_0^{2\pi}\int_1^{\sqrt{2}}\int_{-1}^{2}\left(\frac{2r\cos\theta}{r^2} - \frac{2z}{r^2} + r\right) dz\, r\, dr\, d\theta = \int_0^{2\pi}\int_1^{\sqrt{2}}\left(6\cos\theta - \frac{3}{r} + 3r^2\right) dr\, d\theta$
$-\int_0^{2\pi}\left[6\left(\sqrt{2}-1\right)\cos\theta - 3\ln\sqrt{2} + 2\sqrt{2} - 1\right] d\theta = 2\pi\left(-\frac{3}{2}\ln 2 + 2\sqrt{2} - 1\right)$

Exercise 6.

(a) *Show that if the necessary partial derivatives of the components of the field* $\mathsf{G} = M\mathsf{i} + N\mathsf{j} + P\mathsf{k}$ *are continuous, then* $\nabla . \nabla \times \mathsf{G} = 0$.

(b) *What, if anything, can you conclude about the flux of the field* $\nabla \times \mathsf{G}$ *across a closed surface? Give reason for your answer.*

Solution for Exercise 6

(a) $G = M\mathbf{i} + N\mathbf{j} + P\mathbf{k} \Rightarrow \nabla \times G = curlG =$
$\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\mathbf{k} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k} \Rightarrow \nabla \cdot \nabla \times G$
$= div(curl\mathbf{G}) = \frac{\partial}{\partial x}\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$
$= \frac{\partial^2 P}{\partial x \partial y} - \frac{\partial^2 N}{\partial x \partial z} + \frac{\partial^2 M}{\partial y \partial z} - \frac{\partial^2 P}{\partial y \partial x} + \frac{\partial^2 N}{\partial z \partial x} - \frac{\partial^2 M}{\partial z \partial y} = 0$ if all first and second partial derivatives are continuous

(b) By the Divergence Theorem, the outward flux of $\nabla \times \mathbf{G}$ across a closed surface is zero because outward flux of $\nabla \times \mathbf{G} = \iint_S (\nabla \times \mathbf{G}) \cdot \mathbf{n} d\sigma$
$= \iiint_D \nabla \cdot \nabla \times \mathbf{G} dV$ [Divergence Theorem with $\mathbf{F} = \nabla \times \mathbf{G}$]
$= \iiint_D (0) dV = 0$ [by part (a)]

Exercise 7.

Let $\mathbf{F}_1$ *and* $\mathbf{F}_2$ *be differentiable vector field and let a and b be arbitrary real constants. Verify the following identities.*

(a) $\nabla \cdot (a\mathbf{F}_1 + b\mathbf{F}_2) = a\nabla \cdot \mathbf{F}_1 + b\nabla \cdot \mathbf{F}_2$

(b) $\nabla \times (a\mathbf{F}_1 + b\mathbf{F}_2) = a\nabla \times \mathbf{F}_1 + b\nabla \times \mathbf{F}_2$

(c) $\nabla \cdot (\mathbf{F}_1 \times \mathbf{F}_2) = \mathbf{F}_2 \cdot \nabla \times \mathbf{F}_1 - \mathbf{F}_1 \cdot \nabla \times \mathbf{F}_2$

Solution for Exercise 7

(a) Let $\mathbf{F}_1 = M_1\mathbf{i} + N_1\mathbf{j} + P_1\mathbf{k}$ and $\mathbf{F}_2 = M_2\mathbf{i} + N_2\mathbf{j} + P_2\mathbf{k} \Rightarrow a\mathbf{F}_1 + b\mathbf{F}_2$
$= (aM_1 + bM_2)\mathbf{i} + (aN_1 + bN_2)\mathbf{j} + (aP_1 + bP_2)\mathbf{k} \Rightarrow \nabla \cdot (a\mathbf{F}_1 + b\mathbf{F}_2)$
$= \left(a\frac{\partial M_1}{\partial x} + b\frac{\partial M_2}{\partial x}\right) + \left(a\frac{\partial N_1}{\partial y} + b\frac{\partial N_2}{\partial y}\right) + \left(a\frac{\partial P_1}{\partial z} + b\frac{\partial P_2}{\partial z}\right)$
$= a\left(\frac{\partial M_1}{\partial x} + \frac{\partial N_1}{\partial y} + \frac{\partial P_1}{\partial z}\right) + b\left(\frac{\partial M_2}{\partial x} + \frac{\partial N_2}{\partial y} + \frac{\partial P_2}{\partial z}\right) = a(\nabla \cdot \mathbf{F}_1) + b(\nabla \cdot \mathbf{F}_2)$

(b) Define $\mathbf{F}_1$ and $\mathbf{F}_2$ as in part a $\Rightarrow \nabla \times (a\mathbf{F}_1 + b\mathbf{F}_2)$
$= \left[\left(a\frac{\partial P_1}{\partial y} + b\frac{\partial P_2}{\partial y}\right) - \left(a\frac{\partial N_1}{\partial z} + b\frac{\partial N_2}{\partial z}\right)\right]\mathbf{i} + \left[\left(a\frac{\partial M_1}{\partial z} + b\frac{\partial M_2}{\partial z}\right) - \left(a\frac{\partial P_1}{\partial x} + b\frac{\partial P_2}{\partial x}\right)\right]\mathbf{j} +$
$\left[\left(a\frac{\partial N_1}{\partial x} + b\frac{\partial N_2}{\partial x}\right) - \left(a\frac{\partial M_1}{\partial y} + b\frac{\partial M_2}{\partial y}\right)\right]\mathbf{k} = a\left[\left(\frac{\partial P_1}{\partial y} - \frac{\partial N_1}{\partial z}\right)\mathbf{i} + \left(\frac{\partial M_1}{\partial z} - \frac{\partial P_1}{\partial x}\right)\mathbf{j} + \left(\frac{\partial N_1}{\partial x} - \frac{\partial M_1}{\partial y}\right)\mathbf{k}\right] +$
$b\left[\left(\frac{\partial P_2}{\partial y} - \frac{\partial N_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial M_2}{\partial z} - \frac{\partial P_2}{\partial x}\right)\mathbf{j} + \left(\frac{\partial N_2}{\partial x} - \frac{\partial M_2}{\partial y}\right)\mathbf{k}\right] = a\nabla \times \mathbf{F}_1 + b\nabla \times \mathbf{F}_2$

(c) $\mathbf{F}_1 \times \mathbf{F}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ M_1 & N_1 & P_1 \\ M_2 & N_2 & P_2 \end{vmatrix} = (N_1P_2 - P_1N_2)\mathbf{i} - (M_1P_2 - P_1M_2)\mathbf{j} + (M_1N_2 - N_1M_2)\mathbf{k} \Rightarrow \nabla \cdot (\mathbf{F}_1 \times \mathbf{F}_2)$
$= \nabla \cdot [(N_1P_2 - P_1N_2)\mathbf{i} - (M_1P_2 - P_1M_2)\mathbf{j} + (M_1N_2 - N_1M_2)\mathbf{k}]$
$= \frac{\partial}{\partial x}(N_1P_2 - P_1N_2) - \frac{\partial}{\partial y}(M_1P_2 - P_1M_2) + \frac{\partial}{\partial z}(M_1N_2 - N_1M_2) = \left(P_2\frac{\partial N_1}{\partial x} + N_1\frac{\partial P_2}{\partial x} - N_2\frac{\partial P_1}{\partial x} - P_1\frac{\partial N_2}{\partial x}\right) -$
$\left(M_1\frac{\partial P_2}{\partial y} + P_2\frac{\partial M_1}{\partial y} - P_1\frac{\partial M_2}{\partial y} - M_2\frac{\partial P_1}{\partial y}\right) + \left(M_1\frac{\partial N_2}{\partial z} + N_2\frac{\partial M_1}{\partial z} - N_1\frac{\partial M_2}{\partial z} - M_2\frac{\partial N_1}{\partial z}\right)$
$= M_2\left(\frac{\partial P_1}{\partial y} - \frac{\partial N_1}{\partial z}\right) + N_2\left(\frac{\partial M_1}{\partial z} - \frac{\partial P_1}{\partial x}\right) + P_2\left(\frac{\partial N_1}{\partial x} - \frac{\partial M_1}{\partial y}\right) + M_1\left(\frac{\partial N_2}{\partial z} - \frac{\partial P_2}{\partial y}\right) + N_1\left(\frac{\partial P_2}{\partial x} - \frac{\partial M_2}{\partial z}\right) +$
$P_1\left(\frac{\partial M_2}{\partial y} - \frac{\partial N_2}{\partial x}\right) = \mathbf{F}_2 \cdot \nabla \times \mathbf{F}_1 - \mathbf{F}_1 \cdot \nabla \times \mathbf{F}_2$

Exercise 8.

Let F *be a differentiable vector field and let* $g(x, y, z)$ *be a differentiable scalar function. Verify the following identities.*

(a) $\nabla \cdot (g\mathsf{F}) = g\nabla \cdot \mathsf{F} + \nabla g \cdot \mathsf{F}$

(b) $\nabla \times (g\mathsf{F}) = g\nabla \times \mathsf{F} + \nabla g \times \mathsf{F}$

Solution for Exercise 8

(a) $div(g\mathbf{F}) = \nabla \cdot g\mathbf{F} = \frac{\partial}{\partial x}(gM) + \frac{\partial}{\partial y}(gN) + \frac{\partial}{\partial z}(gP) = \left(g\frac{\partial M}{\partial x} + M\frac{\partial g}{\partial x}\right) + \left(g\frac{\partial N}{\partial y} + N\frac{\partial g}{\partial y}\right) + \left(g\frac{\partial P}{\partial z} + P\frac{\partial g}{\partial z}\right)$
$= \left(M\frac{\partial g}{\partial x} + N\frac{\partial g}{\partial y} + P\frac{\partial g}{\partial z}\right) + g\left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial p}{\partial z}\right) = g\nabla \cdot \mathbf{F} + \nabla g \cdot \mathbf{F}$

(b) $\nabla \times (g\mathbf{F}) = \left[\frac{\partial}{\partial y}(gP) - \frac{\partial}{\partial z}(gN)\right]\mathbf{i} + \left[\frac{\partial}{\partial z}(gM) - \frac{\partial}{\partial x}(gP)\right]\mathbf{j} + \left[\frac{\partial}{\partial x}(gN) - \frac{\partial}{\partial y}(gM)\right]\mathbf{k}$
$= \left(P\frac{\partial g}{\partial y} + g\frac{\partial P}{\partial y} - N\frac{\partial g}{\partial z} - g\frac{\partial N}{\partial z}\right)\mathbf{i} + \left(M\frac{\partial g}{\partial z} + g\frac{\partial M}{\partial z} - P\frac{\partial g}{\partial x} - g\frac{\partial P}{\partial x}\right)\mathbf{j} + \left(N\frac{\partial g}{\partial x} + g\frac{\partial N}{\partial x} - M\frac{\partial g}{\partial y} - g\frac{\partial M}{\partial y}\right)\mathbf{k}$
$= \left(P\frac{\partial g}{\partial y} - N\frac{\partial g}{\partial z}\right)\mathbf{i} + \left(g\frac{\partial P}{\partial y} - g\frac{\partial N}{\partial z}\right)\mathbf{i} + \left(M\frac{\partial g}{\partial z} - P\frac{\partial g}{\partial x}\right)\mathbf{j} + \left(g\frac{\partial M}{\partial z} - g\frac{\partial P}{\partial x}\right)\mathbf{j} + \left(N\frac{\partial g}{\partial x} - M\frac{\partial g}{\partial y}\right)\mathbf{k} +$
$\left(g\frac{\partial N}{\partial x} - g\frac{\partial M}{\partial y}\right)\mathbf{k} = g\nabla \times \mathbf{F} + \nabla g \times \mathbf{F}$

Exercise 9.

If $\mathrm{F} = M\mathrm{i} + N\mathrm{j} + P\mathrm{k}$ *is a differentiable vector field, we define the notation* $\mathrm{F} \cdot \nabla$ *to mean*

$$M\frac{\partial}{\partial x} + N\frac{\partial}{\partial y} + P\frac{\partial}{\partial z}.$$

For differentiable vector fields F_1 *and* F_2*, verify the following identities.*

(a) $\nabla \times (\mathrm{F}_1 \times \mathrm{F}_2) = (\mathrm{F}_2 \cdot \nabla)\mathrm{F}_1 - (\mathrm{F}_1.\nabla)\mathrm{F}_2 + (\nabla \cdot \mathrm{F}_2)\mathrm{F}_1 - (\nabla \cdot \mathrm{F}_1)\mathrm{F}_2$

(b) $\nabla(\mathrm{F}_1 \cdot \mathrm{F}_2) = (\mathrm{F}_1 \cdot \nabla)\mathrm{F}_2 + (\mathrm{F}_2.\nabla)\mathrm{F}_1 + \mathrm{F}_1 \times (\nabla \times \mathrm{F}_2) + \mathrm{F}_2 \times (\nabla \times \mathrm{F}_1)$

Solution for Exercise 9

Let $\mathbf{F}_1 = M_1\mathbf{i} + N_1\mathbf{j} + P_1\mathbf{k}$ and $\mathbf{F}_2 = M_2\mathbf{i} + N_2\mathbf{j} + P_2\mathbf{k}$

(a) $\mathbf{F}_1 \times \mathbf{F}_2 = (N_1P_2 - P_1N_2)\mathbf{i} + (P_1M_2 - M_1P_2)\mathbf{j} + (M_1N_2 - N_1M_2)\mathbf{k} \Rightarrow \nabla \times (\mathbf{F}_1 \times \mathbf{F}_2)$
$= \left[\frac{\partial}{\partial y}(M_1N_2 - N_1M_2) - \frac{\partial}{\partial z}(P_1M_2 - M_1P_2)\right]\mathbf{i} + \left[\frac{\partial}{\partial z}(N_1P_2 - P_1N_2) - \frac{\partial}{\partial x}(M_1N_2 - N_1M_2)\right]\mathbf{j} +$
$\left[\frac{\partial}{\partial x}(P_1M_2 - M_1P_2) - \frac{\partial}{\partial y}(N_1P_2 - P_1N_2)\right]\mathbf{k}$
and consider the $\mathbf{i}$-component only: $\frac{\partial}{\partial y}(M_1N_2 - N_1M_2) - \frac{\partial}{\partial z}(P_1M_2 - M_1P_2)$
$= N_2\frac{\partial M_1}{\partial y} + M_1\frac{\partial N_2}{\partial y} - M_2\frac{\partial N_1}{\partial y} - N_1\frac{\partial M_2}{\partial y} - M_2\frac{\partial P_1}{\partial z} - P_1\frac{\partial M_2}{\partial z} + P_2\frac{\partial M_1}{\partial z} + M_1\frac{\partial P_2}{\partial z}$
$= \left(N_2\frac{\partial M_1}{\partial y} + P_2\frac{\partial M_1}{\partial z}\right) - \left(N_1\frac{\partial M_2}{\partial y} + P_1\frac{\partial M_2}{\partial z}\right) + \left(\frac{\partial N_2}{\partial y} + \frac{\partial P_2}{\partial z}\right)M_1 - \left(\frac{\partial N_1}{\partial y} + \frac{\partial P_1}{\partial z}\right)M_2$
$= \left(M_2\frac{\partial M_1}{\partial x} + N_2\frac{\partial M_1}{\partial y} + P_2\frac{\partial M_1}{\partial z}\right) - \left(M_1\frac{\partial M_2}{\partial x} + N_1\frac{\partial M_2}{\partial y} + P_1\frac{\partial M_2}{\partial z}\right) + \left(\frac{\partial M_2}{\partial x} + \frac{\partial N_2}{\partial y} + \frac{\partial P_2}{\partial z}\right)M_1 -$
$\left(\frac{\partial M_1}{\partial x} + \frac{\partial N_1}{\partial y} + \frac{\partial P_1}{\partial z}\right)M_2$. Now, $\mathbf{i}$-comp of $(\mathbf{F}_2 \cdot \nabla)\mathbf{F}_1 = \left(M_2\frac{\partial}{\partial x} + N_2\frac{\partial}{\partial y} + P_2\frac{\partial}{\partial z}\right)M_1$
$= \left(M_2\frac{\partial M_1}{\partial x} + N_2\frac{\partial M_1}{\partial y} + P_2\frac{\partial M_1}{\partial z}\right)$; likewise, $\mathbf{i}$-comp of $(\mathbf{F}_1 \cdot \nabla)\mathbf{F}_2 = \left(M_1\frac{\partial M_2}{\partial x} + N_1\frac{\partial M_2}{\partial y} + P_1\frac{\partial M_2}{\partial z}\right)$; $\mathbf{i}$-comp of
$(\nabla \cdot \mathbf{F}_2)\mathbf{F}_1 = \left(\frac{\partial M_2}{\partial x} + \frac{\partial N_2}{\partial y} + \frac{\partial P_2}{\partial z}\right)M_1$ and $\mathbf{i}$-comp of $(\nabla \cdot \mathbf{F}_1)\mathbf{F}_2 = \left(\frac{\partial M_1}{\partial x} + \frac{\partial N_1}{\partial y} + \frac{\partial P_1}{\partial z}\right)M_2$.
Similar results hold for the $\mathbf{j}$ and $\mathbf{k}$ components of $\nabla \times (\mathbf{F}_1 \times \mathbf{F}_2)$. In summary, since the corresponding components are equal,we have the result
$\nabla \times (\mathbf{F}_1 \times \mathbf{F}_2) = (\mathbf{F}_2 \cdot \nabla)\mathbf{F}_1 - (\mathbf{F}_1 \cdot \nabla)\mathbf{F}_2 + (\nabla \cdot \mathbf{F}_2)\mathbf{F}_1 - (\nabla \cdot \mathbf{F}_1)\mathbf{F}_2$

(b) Here again we consider only the $\mathbf{i}$-component of each expression. Thus, the $\mathbf{i}$-comp of
$\nabla(\mathbf{F}_1 \cdot \mathbf{F}_2) = \frac{\partial}{\partial x}(M_1M_2 + N_1N_2 + P_1P_2) = \left(M_1\frac{\partial M_2}{\partial x} + M_2\frac{\partial M_1}{\partial x} + N_1\frac{\partial N_2}{\partial x} + N_2\frac{\partial N_1}{\partial x} + P_1\frac{\partial P_2}{\partial x} + P_2\frac{\partial P_1}{\partial x}\right)$,
$\mathbf{i}$-comp of $(\mathbf{F}_1 \cdot \nabla)\mathbf{F}_2 = \left(M_1\frac{\partial M_2}{\partial x} + N_1\frac{\partial M_2}{\partial y} + P_1\frac{\partial M_2}{\partial z}\right)$, $\mathbf{i}$-comp of $(\mathbf{F}_2 \cdot \nabla)\mathbf{F}_1 = \left(M_2\frac{\partial M_1}{\partial x} + N_2\frac{\partial M_1}{\partial y} + P_2\frac{\partial M_1}{\partial z}\right)$,
$\mathbf{i}$-comp of $\mathbf{F}_1 \times (\nabla \times \mathbf{F}_2) = N_1\left(\frac{\partial N_2}{\partial x} - \frac{\partial M_2}{\partial y}\right) - P_1\left(\frac{\partial M_2}{\partial z} - \frac{\partial P_2}{\partial x}\right)$, and
$\mathbf{i}$-comp of $\mathbf{F}_2 \times (\nabla \times \mathbf{F}_1) = N_2\left(\frac{\partial N_1}{\partial x} - \frac{\partial M_1}{\partial y}\right) - P_2\left(\frac{\partial M_1}{\partial z} - \frac{\partial P_1}{\partial x}\right)$. Since corresponding components are equal, we see that $\nabla(\mathbf{F}_1 \cdot \mathbf{F}_2) = (\mathbf{F}_1 \cdot \nabla)\mathbf{F}_2 + (\mathbf{F}_2 \cdot \nabla)\mathbf{F}_1 + \mathbf{F}_1 \times (\nabla \times \mathbf{F}_2) + \mathbf{F}_2 \times (\nabla \times \mathbf{F}_1)$, as claimed.

Exercise 10.

1. *Let* F *be a field whose components have continuous first partial derivatives throughout a portion of space containing a region* D *bounded by a smooth closed surface* S. *If* $|\mathrm{F}| \leq 1$, *can any bound be placed on the size of*

$$\iiint_D \nabla \cdot \mathrm{F}\, dV?$$

   Give reasons for your answer.

2. *The base of the closed cubelike surface shown here is the unit square in the* xy*-plane. The four sides lie in the planes* $x = 0$, $x = 1$, $y = 0$, *and* $y = 1$. *The top is an arbitrary smooth surface whose identity is unknown. Let* $\mathrm{F} = x\mathrm{i} - 2y\mathrm{j} + (z+3)\mathrm{k}$ *and suppose the outward flux of* F *through Side A is 1 and through Side B is -3. Can you conclude anything about the outward flux through the top? Give reasons for your answer.*

Solution for Exercise 10

1. The integral's value never exceeds the surface area of S. Since $|\mathbf{F}| \leq 1$, we have $|\mathbf{F} \cdot \mathbf{n}| = |\mathbf{F}||\mathbf{n}| \leq (1)(1) = 1$ and

$$\begin{aligned}
\iiint_D \nabla \cdot \mathbf{F} d\sigma &= \iint_S \mathbf{F} \cdot \mathbf{n} d\sigma && \text{[Divergence Theorem]} \\
&\leq \iint_S |\mathbf{F} \cdot \mathbf{n}| d\sigma && \text{[A property of integrals]} \\
&\leq \iint_S (1) d\sigma && [|\mathbf{F} \cdot \mathbf{n}| \leq 1] \\
&= \text{Area of S.}
\end{aligned}$$

2. Yes, the outward flux through the top in 5. The reason is this: Since $\nabla \cdot \mathbf{F} = \nabla \cdot (x\mathbf{i} - 2y\mathbf{j} + (z+3)\mathbf{k}) = 1 - 2 + 1 = 0$, the outward flux across the closed cubelike surface is 0 by the Divergence Theorem. The flux across the top is therefore the negative of the flux across the sides and base. Routine calculations show that the sum of these latter fluxes is -5. (The flux across the sides that lie in the xz-plane and the yz-plane are 0, while the flux across the xy-plane is -3.) Therefore the flux across the top is 5.

Exercise 11.

1. (a) *Show that the outward flux of the position vector field* $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ *through a smooth closed surface* S *is three times the volume of the region enclosed by the surface.*

   (b) *Let* $\mathbf{n}$ *be the outward unit normal vector field on* S. *Show that it is not possible for* $\mathbf{F}$ *to be orthogonal to* $\mathbf{n}$ *at every point of* S.

2. Maximum flux *Among all rectangular solids defined by the inequalities* $0 \leq x \leq a$, $0 \leq y \leq b$, $0 \leq z \leq 1$, *find the one for which the total flux of* $\mathbf{F} = (-x^2 - 4xy)\mathbf{i} - 6yz\mathbf{j} + 12z\mathbf{k}$ *out-ward through the six sides is greatest. What is the greatest flux?*

3. Volume of a solid region : *Let* $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ *and suppose that the surface* S *and region* D *satisfy the hypotheses of the Divergence Theorem. Show that the volume of* D *is given by the formula*

$$\text{Volume of } D = \frac{1}{3} \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma.$$

4. Outward flux of a constant field : *Show that the outward flux of a constant vector field* $\mathbf{F} = \mathbf{C}$ *across any closed surface to which the Divergence Theorem applies is zero.*

Solution for Exercise 11

1. (a) $\frac{\partial}{\partial x}(x) = 1,\ \frac{\partial}{\partial y}(y) = 1,\ \frac{\partial}{\partial z}(z) = 1 \Rightarrow \nabla \cdot \mathbf{F} = 3 \Rightarrow$ Flux $= \iiint\limits_D 3dV = 3 \iiint\limits_D dV = 3$ (Volume of the solid)

   (b) If $\mathbf{F}$ is orthogonal to $\mathbf{n}$ at every point of S, then $\mathbf{F} \cdot \mathbf{n} = 0$ everywhere $\Rightarrow$ Flux $= \iint\limits_S \mathbf{F} \cdot \mathbf{n} d\sigma = 0$. But the flux is 3 (Volume of the solid)$\neq 0$, so $\mathbf{F}$ is not orthogonal to $\mathbf{n}$ at every point.

2. $\nabla \cdot \mathbf{F} = -2x - 4y - 6z + 12 \Rightarrow$ Flux $= \int_0^a \int_0^b \int_0^1 (-2x - 4y - 6z + 12) dzdydx = \int_0^a \int_0^b (-2x - 4y + 9) dydx = \int_0^a (-2xb - 2b^2 + 9b) dx = -a^2 b - 2ab^2 + 9ab = ab(-a - 2b + 9) = f(a, b)$; $\frac{\partial f}{\partial a} = -2ab - 2b^2 + 9b$ and $\frac{\partial f}{\partial b} = -a^2 - 4ab + 9a$ so that $\frac{\partial f}{\partial a} = 0$ and $\frac{\partial f}{\partial b} = 0 \Rightarrow b(-2a - 2b + 9) = 0$ and $a(-a - 4b + 9) = 0 \Rightarrow b = 0$ or $-2a - 2b + 9 = 0$, and $a = 0$ or $-a - 4b + 9 = 0$. Now $b = 0$ or $a = 0 \Rightarrow$ Flux $= 0$; $-2a - 2b + 9 = 0$ and $-a - 4b + 9 = 0 \Rightarrow 3a - 9 = 0 \Rightarrow a = 3 \Rightarrow b = \frac{3}{2}$ so that $f(3, \frac{3}{2}) = \frac{27}{2}$ is the maximum flux.

3. $\iint\limits_S \mathbf{F} \cdot \mathbf{n} d\sigma = \iiint\limits_D \nabla \cdot \mathbf{F} dV = \iiint\limits_D 3dV \Rightarrow \frac{1}{3} \iint\limits_S \mathbf{F} \cdot \mathbf{n} d\sigma = \iiint\limits_D dV =$ Volume of D

4. $\mathbf{F} = \mathbf{C} \Rightarrow \nabla \cdot \mathbf{F} = 0 \Rightarrow$ Flux $= \iint\limits_S \mathbf{F} \cdot \mathbf{n} d\sigma = \iiint\limits_D \nabla \cdot \mathbf{F} dV = \iiint\limits_D 0 dV = 0$

Exercise 12.

1. Harmonic functions : *A function $f(x, y, z)$ is said to be harmonic in a region D in space if it satisfies the Laplace equation*

$$\nabla^2 f = \nabla \cdot \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

throughout D.

   (a) *Suppose that f is harmonic throughout a bounded region D enclosed by a smooth surface S and that* n *is the chosen unit normal vector on S. Show that the integral over S of $\nabla f \cdot$ n, the derivative of f in the direction of* n, *is zero.*

   (b) *Show that if f is harmonic on D, then*

$$\iint_S f \nabla f \cdot \mathrm{n}\, d\sigma = \iiint_D |\nabla f|^2\, dV.$$

2. Outward flux of a gradient field : *Let S be the surface of the portion of the solid sphere $x^2 + y^2 + z^2 \leq a^2$ that lies in the first octant and let $f(x, y, z) = \ln \sqrt{x^2 + y^2 + z^2}$. Calculate*

$$\iint \nabla f \cdot \mathrm{n}\, d\sigma$$

Solution for Exercise 12

1. (a) From the Divergence Theorem,
   $\iint\limits_S \nabla f \cdot \mathbf{n} d\sigma = \iiint\limits_D \nabla \cdot \nabla f dV = \iiint\limits_D \nabla^2 f dV = \iiint\limits_D 0 dV = 0.$

   (b) From the divergence Theorem, $\iint\limits_S f\nabla f \cdot \mathbf{n} d\sigma = \iiint\limits_D \nabla \cdot f \nabla f dV$. Now,
   $f\nabla f = \left(f\frac{\partial f}{\partial x}\right)\mathbf{i} + \left(f\frac{\partial f}{\partial y}\right)\mathbf{j} + \left(f\frac{\partial f}{\partial z}\right)\mathbf{k} \Rightarrow \nabla \cdot f\nabla f =$
   $\left[f\frac{\partial^2 f}{\partial x^2} + \left(\frac{\partial f}{\partial x}\right)^2\right] + \left[f\frac{\partial^2 f}{\partial y^2} + \left(\frac{\partial f}{\partial y}\right)^2\right] + \left[f\frac{\partial^2 f}{\partial z^2} + \left(\frac{\partial f}{\partial z}\right)^2\right]$
   $= f\nabla^2 f + |\nabla f|^2 = 0 + |\nabla f|^2$ since f is harmonic $\Rightarrow \iint\limits_S f\nabla f \cdot \mathbf{n} d\sigma = \iiint\limits_D |\nabla f|^2 dV$,
   as claimed.

2. From the Divergence Theorem,
   $\iint\limits_S \nabla f \cdot \mathbf{n} d\sigma = \iiint\limits_D \nabla \cdot \nabla f dV = \iiint\limits_D \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right) dV$. Now,
   $f(x,y,z) = \ln\sqrt{x^2+y^2+z^2} = \frac{1}{2}\ln(x^2+y^2+z^2) \Rightarrow \frac{\partial f}{\partial x} = \frac{x}{x^2+y^2+z^2}, \frac{\partial f}{\partial y} = \frac{y}{x^2+y^2+z^2},$
   $\frac{\partial f}{\partial z} = \frac{z}{x^2+y^2+z^2}$
   $\Rightarrow \frac{\partial^2 f}{\partial x^2} = \frac{-x^2+y^2+z^2}{(x^2+y^2+z^2)^2}, \frac{\partial^2 f}{\partial y^2} = \frac{x^2-y^2+z^2}{(x^2+y^2+z^2)^2}, \frac{\partial^2 f}{\partial z^2} = \frac{x^2+y^2-z^2}{(x^2+y^2+z^2)^2}, \Rightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$
   $= \frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^2} = \frac{1}{x^2+y^2+z^2} \Rightarrow \iint\limits_S \nabla f \cdot \mathbf{n} d\sigma = \iiint\limits_D \frac{dV}{x^2+y^2+z^2} = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^a \frac{\rho^2 \sin\phi}{\rho^2} d\rho d\phi d\theta$
   $= \int_0^{\pi/2}\int_0^{\pi/2} a\sin\phi d\phi d\theta = \int_0^{\pi/2} [-a\cos\phi]_0^{\pi/2} d\theta = \int_0^{\pi/2} a d\theta = \frac{\pi a}{2}.$

Exercise 13.

1. Green's first formula *Suppose that f and g are scalar functions with continuous first- and second-order partial derivatives throughout a region D that is bounded by a closed piecewise smooth surface S. Show that*

$$\iint_S f\nabla g \cdot \mathrm{n}\, d\sigma = \iiint_D (f\nabla^2 g + \nabla f \cdot \nabla g)\, dV. \tag{1}$$

Equation (1) is Green's first formula*. (Hint: Apply the Divergence Theorem to the field* $\mathrm{F} = f\nabla g$*.)*

2. Green's second formula *(Continuation of Exercise 25.) Interchange f and g in Equation (1) to obtain a similar formula. Then subtract this formula from Equation (1) to show that*

$$\iint_S (f\nabla g - g\nabla f) \cdot \mathrm{n}\, d\sigma = \iiint_D (f\nabla^2 g - g\nabla^2 f)\, dV. \tag{2}$$

This equation is Green's second formula*.*

Solution for Exercise 13

1. $\iint\limits_S f\nabla g \cdot \mathbf{n} d\sigma = \iiint\limits_D \nabla \cdot f\nabla g dV = \iiint\limits_D \nabla \cdot \left(f\frac{\partial g}{\partial x}\mathbf{i} + f\frac{\partial g}{\partial y}\mathbf{j} + f\frac{\partial g}{\partial z}\mathbf{k} \right) dV$
$= \iiint\limits_D \left(f\frac{\partial^2 g}{\partial x^2} + \frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + f\frac{\partial^2 g}{\partial y^2} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial y} + f\frac{\partial^2 g}{\partial z^2} + \frac{\partial f}{\partial z}\frac{\partial g}{\partial z} \right) dV$
$= \iiint\limits_D \left[f\left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2} \right) + \left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial g}{\partial z} \right) \right] dV =$
$\iiint\limits_D \left(f\nabla^2 g + \nabla f \cdot \nabla g \right) dV$

2. By the above exercise, $\iint\limits_S f\nabla g \cdot \mathbf{n} d\sigma = \iiint\limits_D \left(f\nabla^2 g + \nabla f \cdot \nabla g \right) dV$ and by interchanging the roles of f and g,
$\iint\limits_S g\nabla f \cdot \mathbf{n} d\sigma = \iiint\limits_D \left(g\nabla^2 f + \nabla g \cdot \nabla f \right) dV$. Subtracting the second equation from the first yields:
$\iint\limits_S (f\nabla g - g\nabla f) \cdot \mathbf{n} d\sigma = \iiint\limits_D (f\nabla^2 g - g\nabla^2 f) dV$ since $\nabla f \cdot \nabla g = \nabla g \cdot \nabla f$.

Exercise 14.

Let v(t, x, y, z) *be a continuously differentiable vector field over the region* D *in space and let* $p(t, x, y, z)$ *be a continuously differentiable scalar function. The variable* t *represents the time domain. The Law of Conservation of Mass asserts that*

$$\frac{d}{dt}\iiint\limits_D p(t, x, y, z)\, dV = -\iint\limits_S p\mathrm{v}\cdot \mathrm{n}\, d\sigma,$$

where S *is the surface enclosing* D*.*

(a) *Give a physical interpretation of the conservation of mass law if* v *is a velocity flow field and* p *represents the density of the fluid at point* (x, y, z) *at time* t*.*

(b) *Use the Divergence Theorem and Leibniz's Rule,*

$$\frac{d}{dt}\iiint\limits_D p(t, x, y, z)\, dV = \iiint\limits_D \frac{\partial p}{\partial t}\, dV,$$

to show that the Law of Conservation of Mass is equivalent to the continuity equation,

$$\nabla . p\mathrm{v} + \frac{\partial p}{\partial t} = 0.$$

Solution for Exercise 14

(a) The integral $\iiint\limits_D p(t,x,y,z)dV$ represents the mass of the fluid at any time t. The equation says that the instantaneous rate of change of mass is flux of the fluid through the surface S enclosing the region D: the mass decreases if the flux is outward (so the fluid flows out of D), and increases if the flow is inward (interpreting $\mathbf{n}$ as the outward pointing unit normal to the surface).

(b) $\iiint\limits_D \frac{\partial p}{\partial t} dV = \frac{d}{dt} \iiint\limits_D p dV = -\iint\limits_S p\mathbf{v} \cdot \mathbf{n} d\sigma = -\iiint\limits_D \nabla \cdot p\mathbf{v} dV \Rightarrow \frac{\partial \rho}{\partial t} = -\nabla \cdot p\mathbf{v}$

Since the law is to hold for all regions D, $\nabla \cdot p\mathbf{v} + \frac{\partial p}{\partial t} = 0$, as claimed.

The heat diffusion equation

Exercise 15.

Let $T(t, x, y, z)$ *be a function with continuous second derivatives giving the temperature at time* t *at the point* (x, y, z) *of a solid occupying a region* D *in space. If the solid's heat capacity and mass density are denoted by the constants* c *and* ρ*, respectively, the quantity* $c\rho T$ *is called the solid's* heat energy per unit volume.

(a) *Explain why* $-\nabla T$ *points in the direction of heat flow.*

(b) *Let* $-k\nabla T$ *denote the* energy flux vector*. (Here the constant* k *is called the* conductivity.*) Assuming the Law of Conservation of Mass with* $-k\nabla T = \mathrm{v}$ *and* $c\rho T = p$ *in Exercise 27, derive the diffusion (heat) equation*

$$\frac{\partial T}{\partial t} = K\nabla^2 T,$$

where $K = k/(c\rho) > 0$ *is the diffusivity constant. (Notice that if* $T(t, x)$ *represents the temperature at time* t *at position* x *in a uniform conducting rod with perfectly insulated sides, then* $\nabla^2 T = \partial^2 T/\partial x^2$ *and the diffusion equation reduces to the one-dimensional heat equation in Chapter 14's Additional Exercises.)*

Solution for Exercise 15

(a) ∇T points in the direction of maximum change of the temperature, so if the solid is heating up at the point the temperature is greater in a region surrounding the point $\Rightarrow \nabla T$ points away from the point $\Rightarrow -\nabla T$ points toward the point $\Rightarrow -\nabla T$ points in the direction the heat flows.

(b) Assuming the Law of Conservation of Mass (Exercise 31) with $-k\nabla T = p\mathbf{v}$ and $c\rho T = p$, we have $\frac{d}{dt}\iiint\limits_D c\rho T dV = -\iint\limits_S -k\nabla T \cdot \mathbf{n} d\sigma \Rightarrow$ the continuity equation,

$\nabla \cdot (-k\nabla T) + \frac{\partial}{\partial t}(c\rho T) = 0$

$\Rightarrow c\rho \frac{\partial T}{\partial t} = -\nabla \cdot (-k\nabla T) = k\nabla^2 T \Rightarrow \frac{\partial T}{\partial t} = \frac{k}{c\rho}\nabla^2 T = K\nabla^2 T$, as claimed.

References

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